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NEW DEMONSTRATION AND FORMS OF LAGRANGE'S THEOREM. THE GENERAL THEOREM.

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"In the case of Lagrange's Theorem, Lambert of Alsace (died 1777), in endeavoring to express the roots of Algebraic equations in series, found a law resembling that which we have just developed. Hc published his results in 1758, and Lagrange generalized them into the theorem, which bears his name, in 1772. Finally, in the *Mecanique Celeste*, Laplace made a still further extension." (*DeMorgan's Calculus*, p. 171).

The following investigation from a new point of view, appearing entirely conclusive, is presented to facilitate the study of analysis. At the close, the reasoning unexpectedly leads to the general Theorem, under a simple expoential form, which includes this whole class of Theorems, especially those of Taylor, Lagrange and Laplace, as particular cases, according to the initial differentiations there described.

To find the value of x in terns of t and e from the *primary equation*,

(1) $$x = t + efx = t + h.$$

Let us make the single letter $h = efx$, as already denoted; then taking the same function f of each side,

(2) $$fx = f(t + h).$$

The first member $fx = h \div e$, as indicated above. And this equals the second member developed in powers of h by Taylor's Theorem:

(3) $$\frac{h}{e} = ft + \frac{dft}{dt}.h + \frac{d^2ft}{dt^2}.\frac{h^2}{1.2} + \frac{d^3ft}{dt^3}.\frac{h^3}{1.2.3} + \ldots$$

The auxiliary h can now be found by simple reversion of series; thus let,

(4) $$h = Ae + Be^2 + Ce^3 + De^4 + \ldots .$$

Substituting this series in place of h in equation (3) and equating the coefficients of equal powers of e; also denoting ft by the single letter f,

$$A = ft = f,$$

$$B = \frac{df}{dt} . A = \tfrac{1}{2}\frac{d}{dt}f^2,$$

$$C = \frac{df}{dt} . B + \tfrac{1}{2}\frac{d^2f}{dt^2} . A^2 = \frac{1}{1.2.3} . \frac{d^2}{dt^2}f^3,$$

$$D = \frac{df}{dt} . C + \frac{d^2f}{dt^2} . AB + \tfrac{1}{6}\frac{d^3f}{dt^3} . A^3 = \frac{1}{1.2.3.4} . \frac{d^3f^4}{dt^3}, \dots\dots$$

$$(5) \qquad h = ef + \frac{1}{1.2} . \frac{df^2e^2}{dt} + \frac{1}{1.2.3} . \frac{d^2f^3e^3}{dt^2} + \frac{1}{1.2.3.4} . \frac{d^3f^4e^4}{dt^3} + \dots$$

The regular law of the series continues to any extent, as will be shown presently. Substituting in equation (1) we have a particular case of Lagrange's Theorem:

$$(6) \quad x = t + eft + \frac{1}{1.2} . \frac{d}{dt}\left(eft\right)^2 + \dots + \frac{1}{1.2.3\dots n} . \frac{d^{n-1}}{dt^{n-1}}\left(eft\right)^n + \dots$$

Generally, taking any function of each side of equation (1), and developing by Taylor's Theorem,

$$(7) \qquad Fx = F(t + h) = Ft + \frac{dFt}{dt} . h + \frac{d^2Ft}{dt^2} . \frac{h^2}{1.2} + \dots\dots$$

Here, substituting for h its value from equation (5), and reducing, we have *Lagrange's Theorem:*

$$(8) \; Fx = Ft + eft . \frac{dFt}{dt} + \frac{1}{1.2} . \frac{d}{dt}\left(\overline{eft}^2 . \frac{dFt}{dt}\right) + \frac{1}{1.2.3} . \frac{d^2}{dt^2}\left(\overline{eft}^3 . \frac{dFt}{dt}\right) + \dots$$
$$+ \frac{1}{1.2.3\dots n} . \frac{d^{n-1}}{dt^{n-1}}\left(\overline{eft}^n . \frac{dFt}{dt}\right) + \dots$$

It only remains to prove that the regular law of the first four terms of the series, continues uniform. Developing each member of the last equation by Maclaurin's Theorem, and equating coefficients of equal powers of e,

$$(9) \qquad \left[\frac{d^2Fx}{de^2}\right] = \frac{d}{dt}\left(\overline{ft}^2 . \frac{dFt}{dt}\right); \quad \left[\frac{d^3Fx}{de^3}\right] = \frac{d^2}{dt^2}\left(\overline{ft}^3 . \frac{dFt}{dt}\right).$$

The simple law of third Maclaurin derivation by increasing the exponent of derivation and of ft, each by unity, is obvious. And since it is applicable to any form of the function F, let a new function F' be assumed, such that,

$$(10) \qquad \left[\frac{d^2F'x}{de^2}\right] = \frac{d^2}{dt^2}\left(\overline{ft}^3 . \frac{dFt}{dt}\right) = \left[\frac{d^3Fx}{de^3}\right].$$

Differentiating these equals successively, we see that the law of the advance in Fx must coincide with the law of the rear terms already established in both Fx and Ft; and so on to infinity. Equation (6) is included, by making $Ft = t$.

Note 1. In certain examples, where efx contains t also, the demonstration plainly teaches us to regard such t as constant. That is, to differentiate t only where it has taken the place of x; or rather, to differentiate with respect to x, and change x to t afterward.

Note 2. To derive one part from another, let A_n denote the n^{th} term of the formula (6) or (8) to commence with the second term; then,

$$A_n = \frac{1}{n}\cdot\frac{d}{dt}\Big(eft.A_{n-1}\Big).$$

Example 1. In the regulur application of the formula to equations, Lagrange found that it always gave the least root. Let it then be applied to give the least root of the common quadratic. Here $x = a + bx^2$; by formula (6), $\quad x = a + ba^2 + 2b^2a^3 + 5b^3a^4 + 14b^4a^5 + \ldots.$

The convergence of the series evidently depends on the smallness of ab.

Example 2. It is required to express the third power of the least root of the quintic $x = a + bx^5$.

By formula (8), $Fx = x^3$, $dfx \div dx = 3x^2$, $efx = bx^5$.

$$x^3 = a^3 + 3ba^7 + 18b^2a^{11} + 136b^3a^{15} + \ldots.$$

Example 3. To find what number is equal to ten times its natural or hyperbolic logarithm.

In the primary equation, $t = 0$, $x = 0 + \varepsilon^{.1x}$. By (6),

$$x = 0 + 1 + \frac{0.2}{2!} + \frac{(0.1n)^{n-1}}{n!} + \ldots = 1.118325.$$

Example 4. If a denote the left hand digit or digits of a number and b those of the logarithm, it is required to find the remaining digits such that $a + x = \log(b + .01x)$, or such that the right hand digits of both number and logarithm shall be alike.

Lagrange's Theorem for several variables. Let $efx = h$, $e'f'y = h'$, ... in the primary equations,

(11) $$x = t + efx, \qquad y = t' + e'f'y, \ldots$$

The value of h, of h', ... can each be written out separately by the preceding equation (5). It only remains to substitute these series in the required function (12) after its development in powers of h, h', ... by Taylor's Theorem for several variables:

$$(12)\quad Fx, y, \ldots = Ft+h, t'+h', \ldots = Ft, t', \ldots + \frac{dF}{dt}.h + \frac{dF}{dt'}.h' + \ldots$$
$$+ \frac{d^2F}{dtdt'}.hh' + \tfrac{1}{2}\frac{d^2F}{dt^2}.h^2 + \tfrac{1}{2}\frac{d^2F}{dt'^2}.h'^2 + \ldots$$

The result being independent of the order of operations, is better stated in (26).

Laplace's Theorem. The primary equation may have the more general form,

$$(13)\qquad x = f'(t + efx) = f'(t + h).$$

Taking the function f of each side and developing the powers of h by Taylor's Theorem,

$$(14)\qquad fx = \frac{h}{e} = ff't + \frac{d}{dt}ff't.h + \frac{d^2}{dt^2}ff't.\frac{h^3}{1.2} + \ldots.$$

Let this be compared with equations (3) and (5). Again, taking any function of each side of equation (13) preparatory to development by Taylor's Theorem:

$$(15)\qquad Fx = Ff'(t + h).$$

Let this be compared with equation (7). Whence it is obvious that Laplace's Theorem will be found by simply changing in (8), Ft into $Ff't$, and ft into $ff't$. That is,

$$(16)\qquad Fx = Ff't + eff't.\frac{dFf't}{dt} + \frac{1}{1.2}.\frac{d}{dt}\left(\overline{eff't}^2.\frac{d}{dt}.Ff't\right) + \ldots$$

In the case of several variables, the changes in equations (12) and (26) are similar and obvious, if the primary equations have each but one unknown quantity as in (11).

The more general equations will be investigated presently.

Convergence. To render Lagrange's series more convergent, let a denote any approximate value of $x - t$ or efx; as $a = beft$, where $b = 1 + d(eft) \div dt$, nearly. Then if $t' = t + a$, the primary equation becomes

$$x = t' + (efx - a); \qquad\qquad \text{hence,}$$

$$(17)\quad Fx = Ft' + (eft - a)\frac{dFt'}{dt'} + \frac{1}{1.2}\frac{d}{dt'}\left\{\left(eft' - a\right)^2\frac{dFt'}{dt'}\right\} + \ldots$$

Again, the arbitrary nature of Fx or Ft can generally be made to give a more convergent series. For illustration, let Fx in equation (8) have the form below, where m denotes any assumed positive integer; and $dft = Tdt$,

$$(18)\quad \frac{1}{(fx)^m} = \frac{1}{(ft)^m} - \frac{emT}{(ft)^m} - \frac{e^2}{1.2}.\frac{d}{dt}\left(\frac{mT}{(ft)^{m-1}}\right) - \frac{e^3}{1.2.3}.\frac{d^2}{dt^2}\left(\frac{mT}{(ft)^{m-2}}\right) - \ldots$$
$$- \frac{e^{m+1}}{1.2..(m+1)}.\frac{d^m}{dt^m}\left(mT\right) - \frac{e^{m+2}}{1.2..(m+2)}.\frac{d^{m+1}}{dt^{m+1}}\left(ft.mT\right) - \ldots$$

Observe whenever $T = ft$, one term vanishes by differentiation.

Again, Fx may take the forms below, if integrable:

$$(19)\qquad \int\frac{dx}{fx}=\int\frac{dt}{ft}+e+\frac{e}{1.2}.\frac{d(eft)}{dt}+\frac{e}{1.2.3}.\frac{d^2(eft)^2}{dt^2}+\ldots$$

$$\int\frac{dx}{(fx)^2}=\int\frac{dt}{(ft)^2}+\frac{e}{ft}+0+\frac{e^2}{1.2.3}.\frac{d^2(eft)}{dt^2}+\frac{e^2}{1.2.3.4}.\frac{d^3(eft)^2}{dt^3}+\ldots$$

Example. Kepler's Problem. Applying the last formula to (20) so well known in Astronomy, we obtain (21).

$$(20)\qquad u=t+e\sin u.\qquad \int\frac{du}{\sin^2 u}=-\cot u.$$

$$(21)\qquad \cot u=\cot t-e\div\sin t+\tfrac{1}{6}e^3\sin t+\tfrac{1}{6}e^4\sin 2t-\ldots.$$

Or multiplying by $\sin u\sin t$ and reducing,

$$(22)\qquad \sin(u-t)=e'\sin u.$$

Whence

$$(23)\qquad \tan(u-\tfrac{1}{2}t)=\frac{1+e'}{1-e'}\tan\tfrac{1}{2}t.$$

$$(24)\qquad e'=e-\tfrac{1}{6}e^3\sin^2 t-\tfrac{1}{3}e^4\sin^2 t\cos t+e^6(\tfrac{1}{2}\sin^2 t-\tfrac{27}{40}\sin^4 t)+\ldots$$

The limiting values of this series are remarkably simple. For, taking the sine of each side of (20) after transposing t; multiplying and dividing by $\sin u$; and comparing with (22), we find $\sin(e\sin u)\div\sin u=e'$. Making $\sin u=1$, we have $\sin e=e'$, the least value of e'. Again, making $\sin u$ very small or 0, we find $e=e'$, the greatest value of e'.

Another solution. In (20), and the preceding expression, making $\sin u = 0, 0.1, 0.2, 0.3, \ldots$ the value of e' can also be estimated or interpolated from the following

Outline Table.

$t=$	$u-e\sin u$	$5°44'-0.1e$	$11°32'-0.2e$	$17°27'-0.3e$	$23°35'-0.4e$	
$e'=$	e	$10\sin\frac{1}{10}e$	$\frac{10}{2}\sin\frac{2}{10}e$	$\frac{10}{3}\sin\frac{3}{10}e$	$\frac{10}{4}\sin\frac{4}{10}e$	
$t=$	$30°-0.5e$	$36°52'-0.6e$	$44°26'-0.7e$	$53°08'-0.8e$	$64°10'-0.9e$	$90°-e$
$e'=$	$2\sin\frac{1}{2}e$	$\frac{10}{6}\sin\frac{6}{10}e$	$\frac{10}{7}\sin\frac{7}{10}e$	$\frac{10}{8}\sin\frac{8}{10}e$	$\frac{10}{9}\sin\frac{9}{10}e$	$\sin e$

The value of e is always positive, also that of u, $180°\pm u$, $360°-u$.

Symbolic Terms. Expanding the logarithm below in series, developing each numerator by Taylor's Theorem till the power of the increment h is the same as in the denominator, we see by comparison with (8) that Lagrange's Theorem is represented by the terms independent of h in the development of the following expression:

$$(25)\qquad Fx=Ft-h.\log\left\{1-\frac{ef(t+h)}{h}\right\}.\frac{dF(t+h)}{d(t+h)}.$$

This result is equivalent to the definite integral first found in 1805 by M. Perseval in imaginary exponentials. Again, if the symbol $D=d\div dt$, $D'=d\div dt'$, $D^2=d^2\div dt^2\ldots$ formulas (8) and (12) may also be written as in (26).

The index 1 denotes that the *first* differentiation refers to F only,

$$(26)\quad Fx = \varepsilon^{D_1 eft}.Ft; \qquad Fx, y, \ldots = \varepsilon^{D_1 eft + D'_1 e'f't' + \ldots}.Ft, t', \ldots$$

Integration by Series. Of the two functions contained in Lagrange's full formula, the form of the first, Fx, has thus far been fixed by the primary equation (1), while that of Fx may be assumed at pleasure. Conversely for new forms of development, or of integration by series, let the form of Fx be given, while fx in the primary equation is arbitrary.

Thus let $Fx.dx$ denote any expression to be integrated between the limits x and t. We assume fx to be a constant, or an exponential of x, or a denominator of Fx, or the reciprocal of Fx, or to fulfil any advantageous condition. And since the limits x and t are both known, we find e and eft by the primary equation (1), which gives,

$$(27)\qquad e = \frac{x-t}{fx}, \qquad eft = \frac{(x-t)ft}{fx}.$$

Substituting this value of e in formula (8), also making Fx and Ft in (8) to represent the required integrals here, and transposing the latter integral,

$$(28)\ \int^x Fx\,dx = \left(\frac{x-t}{fx}\right)ft.Ft + \frac{1}{1.2}\left(\frac{x-t}{fx}\right)^2 \frac{d}{dt}\left(\overline{ft}^2.Ft\right) + \frac{1}{1.2.3}\left(\frac{x-t}{fx}\right)^3 \times \frac{d^2}{dt^2}\left(\overline{ft}^3.Ft\right) + \ldots$$

From this general series, a great many others can be derived by assuming the form of the function f. For illustration, let us assume $ft.Ft = 1$, then $fx.Fx = 1$; $(x-t)Fx = e$; whence,

$$(29)\qquad \int_t^x Fxdx = e + \frac{1}{1.2}e^2.\frac{d}{dt}\left(\frac{1}{Ft}\right) + \frac{1}{1.2.3}e^3.\frac{d^2}{dt^2}\left(\frac{1}{Ft}\right)^2 + \ldots$$

For mere development in series without integration, the expression of e in (27) regarded as constant can be substituted in (8), giving what is equivalent to Burmann's Theorem.

For another integral in series, leaving n to be assumed as a whole number or fraction, positive or negative, let

$$ft = Ft^n,\ fx = Fx^n; \text{ then } e = (x-t)Fx^{-n};$$

$$(30)\qquad \int_t^x Fxdx = eFt^{n+1} + \frac{1}{1.2}e^2\frac{d}{dt}.Ft^{2n+1} + \frac{1}{1.2.3}e^3\frac{d^2}{dt^2}.Ft^{3n+1} + \ldots$$

To develop other series of this kind, will afford an interesting class of exercises for students of the Integral Calculus.

The General Theorem.

A review of the preceding pages has unexpectedly led to the more complete formula, toward which the Binomial Theorem and those of Taylor

and Lagrange have progressively tended; the possibility of which was shown by Laplace, with development for one and two variables. The new demonstration appears short, easy and conclusive. Let the primary equations have the form,

(31) $x = t + efx, y, \ldots \qquad y = t' + e'f'x, y, \ldots$ etc.

where $f, f', \ldots$ denote any given functions of $x, y, \ldots$ And let

(32) $h = efxy\ldots;\ h_1 = eftt'\ldots$ Also $h' = e'f'xy\ldots;\ h'_1 = e'f'tt'\ldots$; etc.

Then the law of development of any function F will be

$$(33)\quad Fx, y, \ldots = Ft+h, t'+h'\ldots = \varepsilon^{D_1h_1+D'_1h'_1+\ldots} \times Ft, t', \ldots$$
$$= Ft, t', \ldots + D_1h_1F + \ldots + \frac{D^{n-1}D_1}{n!}.h_1^n F + \ldots + \frac{D^{n-1}D'^{n'-1}\ldots}{n!\,n'!\ldots}$$
$$\times D_1D'_1\ldots h_1^n h_1^{n'}\ldots F + \ldots$$

Here $D_1 = \frac{d}{dt}$; $D_1^2 = \frac{d^2}{dt^2}$; $\ldots D'_1 = \frac{d}{dt'}$; $D_1'^{\,2} = \frac{d^2}{dt'^2}$; etc. The index 1 of D_1 merely denotes that the *first* differentiation is special. Thus $D_1^n = D^{n-1}D_1$, and $D_1^nD_1^{n'} = D^{n-1}D^{n'-1}D_1D'_1$. $Ft, t', \ldots$ in the first term is represented elsewhere by F.

While D^{n-1} is normal, and refers to the whole term before which it is placed, D_1 is special or initial, and refers only to a part of its term, described as follows: When $D_1, D'_1, \ldots$ occur separately, the special differentiation refers to F only. Thus $D_1h_1F = h_1DF$, and $D_1^nh^nF = D^{n-1}(h_1^nDF)$.

But when $D_1, D'_1\ldots$ are multiplied together, we shall have

$$(34)\qquad D_1D'_1 = DD'F + \frac{dh'_1}{dt}D'F + \frac{dh_1}{dt'}DF.$$

As might be inferred from the left member, this notation is not to introduce new factors, but to show the special or initial differentiation of F, h, h', in their place in the given term, which thus becomes three terms.

For i dimensions, with the same signification as to F, h_1, ... the first term is found by the i differentiations of F, as shown below. For the remaining termes, when i is 2 or more, we write the combinations or products of the i quantities $h_1, h'_1, \ldots h^{(i-1)}$ taken $i - 1$ in a set, then $i - 2$ in a set, then $i - 3$, and so on, ending with 1 in a set. After each combination we write D as many times as it had letters, with the same accents respectively, and annexing F, to which these differentiations solely refer. Next, we prefix to each combination and referring to it alone, the complementary number of Ds with such accents, that each term shall contain in all $D, D', D'', \ldots D^{(i-1)}$. Thus for three dimensions,

(35) $D_1D'_1D''_1 = DD'D''F + D''h_1h'_1DD'F + D'h_1h''_1DD''F$
$+ Dh'_1h''_1D'D''F + DD'h''_1D''F + DD''h'_1D'\,F + D'D''h_1DF.$

Note. The result here will be greatly simplified so far as each primary equation is independent of the others, or contains but one varable. Thus if the primary equations all have the form of equations (11) then all the terms containing h in the right hand members of (34), (35) vanish by differentiation leaving only the first term; so that the result is represented by formula (26).

Elimination of the higher Algebraic Equations. When $F = x$, all the terms of (34), (35) vanish except the last, where $DF = 1$; and the first or special derivation of D', D'' etc., must refer to h_1 only. By such reductions we have found from (33) the remarkable result,

$$(36) \qquad x = t + \frac{\varepsilon^{Dh_1}-1}{D}.\varepsilon^{D'_1h'_1+D''_1h''_1+\cdots}.$$

After development, the first differentiation of D'_1, of D''_1, . . . refers to h_1 only; the first of D being already performed. Thus in (33) the exponential of D was expanded separately, and the others together, giving $x = (1 + D_1h_1 + \ldots.)(1 + D'_1h'_1 + D''_1h''_1 + \ldots.)t$; whence (36). Any function of x can be found in like manner.

Another Demonstration of Lagrange's Theorem. Let the primary equation as in (1) be $x = t + efx = t + h$. And let Taylor's Theorem be represented by exponential notation, thus;

$$(37) \qquad Fx = F(\underset{1}{t} + h) = \varepsilon^{\underset{1}{D}h}F\underset{1}{t} = F\underset{1}{t} + \underset{1}{D}F\underset{1}{t}.h + \frac{1}{2\,!}.\underset{1}{D}^2F\underset{1}{t}.h^2 + \ldots$$

$$(38) \qquad h = efx = \varepsilon^{\underset{2}{D}h}.ef\underset{2}{t};\ h^2 = (ef\underset{2}{t})^2.\varepsilon^{\underset{2}{D}h};\ h^n = (ef\underset{2}{t})^n.\varepsilon^{\underset{2}{D}h}.$$

The subscripts 1, 2, . . . will be finally omitted. Their present use is merely to guide the differentiations of Taylor's Theorem; thus $\underset{1}{D}$ applies only to $\underset{1}{t}$; $\underset{2}{D}$ applies only to $\underset{2}{t}$; and so on. While $h = efx$, let $\underset{2}{h} = ef\underset{2}{t}$; $\underset{3}{h} = ef\underset{3}{t}$; etc. By successive substitutions of h from equation (38) to (37),

$$Fx = \varepsilon^{\underset{1}{D}h}F\underset{1}{t} = \varepsilon^{\underset{1}{D}\underset{2}{h}}F\underset{1}{t}\,\varepsilon^{\underset{2}{D}h} = \varepsilon^{\underset{1}{D}\underset{2}{h}+\underset{2}{D}\underset{3}{h}}F\underset{1}{t}\,\varepsilon^{\underset{3}{D}h}.$$

$$(39) \qquad Fx = \varepsilon^{\underset{1}{D}\underset{2}{h}+\underset{2}{D}\underset{3}{h}+\underset{3}{D}\underset{4}{h}+\cdots}\times F\underset{1}{t}.$$

Expanding in series, let the general or n^{th} term be denoted by,

$$(40) \qquad A_n = \frac{1}{n\,!}\Big(\underset{1}{D}\underset{2}{h} + \underset{2}{D}\underset{3}{h} + \ldots\Big)^nF\underset{1}{t} = \frac{1}{n}\Big(\underset{1}{D}\underset{2}{h} + \underset{2}{D}\underset{3}{h} + \ldots\Big)A_{n-1}.$$

As in geometrical progression each term is found from the preceding by a fixed ratio; so here each term is shown to be derivable from the preceding by a fixed operation. It only remains to ascertain or simplify such operation. For this purpose, omitting the subscripts and applying common substitutions from (38) to (37) we readily find

$$A_0 = Ft;$$
$$A_1 = eft.DFt;$$
$$A_2 = \frac{1}{2!}D\left(\overline{eft}^2.DFt\right);$$
$$A_3 = \frac{1}{3!}D^2\left(\overline{eft}^3.DF\right) = \frac{1}{3}D\left(eft.A_2\right);$$
$$A_4 = \frac{1}{4!}D^3\left(\overline{eft}^4.DF\right) = \frac{1}{4}D\left(eft.A_3\right);\ \ldots.$$
$$(41)\qquad A_n = \frac{1}{n!}D^{n-1}\left(\overline{eft}^n.DF\right) = \frac{1}{n}D\left(eft.A_{n-1}\right).$$

Hence the fixed operation in the last member of equation (40) consists in multiplying by $eft \div n$ and taking the derivative of the product. Commencing with A_1, this is the law of the terms in Lagrange's Theorem.

Demonstration of the General Theorem. By Taylor's Theorem, equations (32) become,

$$h = efxy\ldots = eft+h,\ t'+h',\ldots = ef\underset{2}{t}\,\underset{2}{t'}\ldots\varepsilon^{\underset{2}{D}h+\underset{2}{D'}h'+\ldots},$$
$$h^n h'^{n'} = (eft+h,\ t'+h',..)^n(e'f't+h,\ t'+h',..)^{n'} = (ef\underset{2}{t}\,\underset{2}{t'}..)^n(e'f'\underset{2}{t}\,\underset{2}{t'}..)^{n'}\varepsilon^{\underset{2}{D}h+\underset{2}{D'}h'}.$$
$$Fxy\ldots = Ft+h,\ t'+h',\ldots = F\underset{1}{t}\,\underset{1}{t'}\ldots\varepsilon^{\underset{1}{D}h+\underset{1}{D'}h'+\ldots}.$$

In $h, h', \ldots$ of the exponent, substituting $t+h$ for x, $t'+h'$ for $y, \ldots$ and again applying Taylor's Theorem; and so on,

$$Fxy\ldots = F\underset{1}{t}\,\underset{1}{t'}\ldots\varepsilon^{\underset{1}{D}\underset{2}{h}+\underset{1}{D'}\underset{2}{h'}+\ldots}\times\varepsilon^{\underset{2}{D}h+\underset{2}{D'}h'+\ldots};$$
$$(42)\qquad Fxy\ldots = F\underset{1}{t}\,\underset{1}{t'}\ldots\varepsilon^{\Sigma(\underset{i}{D}\underset{i+1}{h}+\underset{i}{D'}\underset{i+1}{h'}+\ldots.)}.$$

Here the summation extends from $i = 1$ to infiinity. Developing in series, and denoting the general term by A_n, we find

$$A_{n+n'+\ldots} = \frac{1}{n!\,n'!\ldots}\left(\Sigma\underset{i}{D}\underset{i+1}{h}\right)^n\left(\Sigma\underset{i}{D'}\underset{i+1}{h'}\right)^{n'}\ldots F\underset{1}{t}\,\underset{1}{t'}\ldots$$
$$= \frac{1}{n}\left(\Sigma\underset{i}{D}\underset{i+1}{h}\right)A_{n-1+n'+\ldots}.$$

To determine this fixed operation, we develop the first terms of formula (42),

and there find it to be always the same as before proved in (40) and (41), except the initial or special differentiation, which was generalized between (34) and (35). This generalization was based on the fixed nature of the operation in connection with the actual developments copied in (34) and (35); which prove the general Theorem, already described.

RECENT RESULTS IN THE STUDY OF LINKAGES.

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COLONEL Peaucellier's discovery of an exact rectilinear motion produced by means of jointed rods is interesting not only for its own sake but as having been the starting point of a new and beautiful branch of Geometrical Study. An account of Peaucellier's invention and of some of the earlier results of the Study of Linkages was contributed by the writer to the ANALYST for March, 1875; the present article relates to more recent results due principly to the English Mathematicians who have worked in this new field.

The term *linkage* is employed by Prof. Sylvester to denote a net-work of jointed bars, such that one bar being fixed, any point of another bar will describe a definite locus while the system changes its shape, or undergoes *deformation*. Thus a jointed quadrilateral is a linkage; if two bars jointed to gether have their free ends jointed one to each of two bars, the system becomes a six-bar linkage. It is evident that in this way we may form a linkage of any *even* number of bars, but that we cannot form a linkage of an odd number of bars.

The joints of a linkage constitute a system of points bound by an even number of conditions, each of which asserts the invariability of the distance of two points of the system. A point rigidly connected with one of the bars, whether on or off the straight line connecting the joints, may be regarded as determined by its distance from the two joints, that is by two conditions of the same form: but when a joint is assumed on a bar already included in the linkage, these conditions are not counted in estimating the order of the linkage, which depends not upon the number of joints but solely upon the number of bars which are moveable relatively to one another. The several joints on a single bar are usually taken in a straight line, but Sylvester remarks that "the true view of the theory of Linkages is to consider each bar as carrying with it an indefinitely extended plane."